Performance Assessment of Constrained Model Predictive Control Systems

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A constrained minimum variance controller is derived based on a moving horizon approach that explicitly accounts for hard constraints on process variables. A procedure for the performance assessment of constrained model predictive control systems is then developed based on the constrained minimum variance controller. The performance bound computed using the proposed moving horizon approach converges to the unconstrained minimum variance performance bound when the constraints on process variables become inactive. The utility of the proposed method in the performance assessment of constrained model predictive control systems is demonstrated through a simulated example.

Introduction

Performance assessment of the model predictive control (MPC) system has been of great interest in both academia and industry due to its practical importance and usage in the chemical process industry. Existing theory for performance assessment, however, does not explicitly handle hard constraints on process variables for MPC control systems (Harris, 1989; Desborough and Harris, 1993; Tyler and Morari, 1996; Harris et al., 1996; Huang et al., 1997; Ko and Edgar, 2000, 2001 Kadali et al., 1999). The presence of constraints renders MPC controller nonlinear and, thus, makes use of traditional linear techniques problematic.

In the literature on the performance assessment of feedback control systems, minimum variance performance bounds have been frequently used as a first-level benchmark performance against which current controller performance can be compared (Harris, 1989; Stanfelj et al., 1993; Harris et al., 1996; Kozub, 1997; Huang et al., 1997; Ko and Edgar, 2000, 2001). These minimum variance performances benchmarks represent theoretically achievable lower bounds on the output variance for linear systems, and, thus, can be useful for distinguishing problems associated with the controller from the problems associated with the process; that is, if a controller is operating at a minimum variance performance bound, further reduction in output variability cannot be achieved through retuning the controller or implementing more advanced controllers. Two survey articles (Qin, 1998; Harris et al., 1999) have been published that give comprehensive overviews of research works up to 1998 on the use of minimum variance performance bounds and their extensions. The use of the minimum variance performance bound as a benchmark is not suitable for processes where constraints on process variables are important, because it does not account for the performance limitation due to constraints on process variables. Depending on the tightness of constraints, the achievable minimum variance performance can be restricted more severely by the presence of constraints than by process time delays. Therefore, it is important to explicitly take the effects of constraints into account to obtain a meaningful minimum variance performance bound.

In this article, a methodology is developed for the estimation of a constrained minimum variance performance bound for multivariable systems with stable process inverses. The performance bound thus obtained can subsequently be used for the performance assessment of the MPC control system as a benchmark of performance. To obtain this, a minimum variance controller is first derived based on a moving horizon approach. In this approach, the minimum mean-square error prediction of the disturbances is used for the optimal predic-

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tion of the outputs over the future horizon. Such minimum variance controller can be designed using a finite-horizon prediction of the outputs. Since the design of the minimum variance controller is based on a moving horizon approach, it can be easily accommodated within the framework of MPC controller formulation. The main utility of the minimum variance controller designed in this way lies in the preassessment of controller performance in which the significance of the effects of constraints is assessed. In the procedure to assess the performance of constrained model predictive control systems using the constrained minimum variance controller, normal operating plant data are used, but knowledge of the process model (such as the step response model) is required to develop the constrained minimum variance controller. The utility of the proposed approach is illustrated using a simulated example.

Predictive Approach for the Design of Minimum Variance Controller

Consider a multivariable control system with m inputs and p outputs that is suitably represented by the following impulse response model forms of the process and the disturbances

$$\begin{aligned} y(t) &= G(q^{-1})u(t) + \Psi(t) \\ &= \sum_{i=1}^{n} H_i q^{-i} u(t) + \sum_{i=0}^{\infty} N_i q^{-i} a(t) \end{aligned} \tag{1}$$

where $G(q^{-1})$ is $p \times m$ process transfer function matrix with H_i as its impulse response coefficient matrices, and $\Psi(t)$ represents disturbance effects on the process outputs; N_i reflects impulse response coefficient matrices of the disturbance model; n is the largest number of time intervals for each output to reach a steady-state threshold; q^{-1} is the backward-shift operator; $a(t)$ represents a white random noise vector of dimension p at discrete-time t; and $y(t)$ and $u(t)$ are the controlled and the manipulated variables that are expressed as deviation variables from set point and steady-state values, respectively. Throughout this article we assume that the process model has a stable inverse.

In this approach we compute a sequence of control inputs at each discrete-time t that minimize the following infinite horizon quadratic cost function of predicted outputs and apply only the first control input computed, that is, a receding horizon approach.

$$J = \lim_{N \to \infty} \frac{1}{N} E\left[\sum_{k=1}^{N} \hat{y}^T(t+k) Q \hat{y}(t+k) \right] \tag{2}$$

where $E[\cdot]$ is the expectation operator; $\hat{y}(t+k)$ is the predicted output at time $t+k$, and Q is a positive-definite symmetric weighting matrix. When the output prediction in Eq. 2 is made in the minimum mean-square error sense, it can be shown that the receding horizon approach above leads to the minimum variance control (see Appendix). To obtain the minimum mean-square error output prediction, we first rewrite the relation given in Eq. 1 to obtain the outputs from time $t+1$ to infinity

$$\begin{aligned}
\begin{bmatrix} y(t+1) \\ y(t+2) \\ y(t+3) \\ \vdots \end{bmatrix} &= \begin{bmatrix} H_1 & 0 & 0 & 0 & \cdots \\ H_2 & H_1 & 0 & 0 & \cdots \\ H_3 & H_2 & H_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \cdot \begin{bmatrix} u(t) \\ u(t+1) \\ u(t+2) \\ \vdots \end{bmatrix} \\
&+ \begin{bmatrix} \sum_{i=1}^{n-1} H_{i+1} u(t-i) \\ \sum_{i=1}^{n-2} H_{i+2} u(t-i) \\ \sum_{i=1}^{n-3} H_{i+3} u(t-i) \\ \vdots \end{bmatrix} + \begin{bmatrix} N_o & 0 & 0 & 0 & \cdots \\ N_1 & N_o & 0 & 0 & \cdots \\ N_2 & N_1 & N_o & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix} \\
&\cdot \begin{bmatrix} a(t+1) \\ a(t+2) \\ a(t+3) \\ \vdots \end{bmatrix} + \begin{bmatrix} \sum_{i=0}^{\infty} N_{i+1} a(t-i) \\ \sum_{i=0}^{\infty} N_{i+2} a(t-i) \\ \sum_{i=0}^{\infty} N_{i+3} a(t-i) \\ \vdots \end{bmatrix}
\end{aligned} \tag{3}$$

or in compact form

$$\lim_{N \to \infty} \bar{y}_N = \lim_{N \to \infty} \left(G_N \bar{u}_N + \tilde{y}_N + D_N \bar{a}_N + \tilde{\Psi}_N \right) \tag{4}$$

where

$$\bar{y}_N = \begin{bmatrix} y(t+1) \\ y(t+2) \\ \vdots \\ y(t+N) \end{bmatrix}, \qquad \bar{u}_N = \begin{bmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+N-1) \end{bmatrix},$$

$$\tilde{y}_N = \begin{bmatrix} \sum_{i=1}^{n-1} H_{i+1} u(t-i) \\ \sum_{i=1}^{n-2} H_{i+2} u(t-i) \\ \vdots \\ \sum_{i=1}^{n-N} H_{i+N} u(t-i) \end{bmatrix}, \qquad \bar{a}_N = \begin{bmatrix} a(t+1) \\ a(t+2) \\ \vdots \\ a(t+N) \end{bmatrix},$$

$$\tilde{\Psi}_N = \begin{bmatrix} \sum_{i=0}^{\infty} N_{i+1} a(t-i) \\ \sum_{i=0}^{\infty} N_{i+2} a(t-i) \\ \vdots \\ \sum_{i=0}^{\infty} N_{i+N} a(t-i) \end{bmatrix},$$

$$G_N = \begin{bmatrix} H_1 & 0 & \cdots & 0 \\ H_2 & H_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ H_N & H_{N-1} & \cdots & H_1 \end{bmatrix},$$

$$D_N = \begin{bmatrix} N_1 & 0 & \cdots & 0 \\ N_2 & N_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ N_N & N_{N-1} & \cdots & N_1 \end{bmatrix}$$

The term $\tilde{\Psi}_N$ in Eq. 4 represents the minimum mean-square error forecasts of the disturbance effects on the process outputs. The forecasted errors, in this case, are given by the term $D_N \bar{a}_N$, which are uncorrelated with $\tilde{\Psi}_N$. The optimal prediction of the output sequence is then given by

$$\begin{bmatrix} \hat{y}(t+1) \\ \hat{y}(t+2) \\ \vdots \end{bmatrix}_{\text{opt}} = G_N \bar{u}_N + \tilde{y}_N + \tilde{\Psi}_N. \tag{5}$$

Substituting the optimal output prediction in Eq. 5 into Eq. 2, the quadratic cost function becomes

$$\begin{aligned} J &= \lim_{N\to\infty} \frac{1}{N} E\left[\sum_{k=1}^{N} \hat{y}^T(t+k) Q \hat{y}(t+k) \right] \\ &= \lim_{N\to\infty} \frac{1}{N} E\left[\left(G_N \bar{u}_N + \tilde{y}_N + \tilde{\Psi}_N \right)^T \bar{Q}_N \left(G_N \bar{u}_N + \tilde{y}_N + \tilde{\Psi}_N \right) \right] \end{aligned} \tag{6}$$

where $\bar{Q}_N$ is an N-block diagonal matrix of Q. The optimal control input sequence that minimizes the objective function in Eq. 6 is given by the following least-squares solution

$$\lim_{N\to\infty} (\bar{u}_N)_{\text{opt}} = -\lim_{N\to\infty} \left[\left(G_N^T \bar{Q}_N G_N \right)^{\dagger} G_N^T \bar{Q}_N \left(\tilde{y}_N + \tilde{\Psi}_N \right) \right] \tag{7}$$

where $(\cdot)^{\dagger}$ denotes the generalized inverse. The minimum variance forecasts $\tilde{\Psi}_N$ in Eq. 7 can be rewritten in terms of $\Psi(t)$ as follows

$$\tilde{\Psi}_N = \begin{bmatrix} \sum_{i=0}^{\infty} N_{i+1} q^{-i} \\ \sum_{i=0}^{\infty} N_{i+2} q^{-i} \\ \vdots \\ \sum_{i=0}^{\infty} N_{i+N} q^{-i} \end{bmatrix} a(t) = \begin{bmatrix} \sum_{i=0}^{\infty} N_{i+1} q^{-i} \\ \sum_{i=0}^{\infty} N_{i+2} q^{-i} \\ \vdots \\ \sum_{i=0}^{\infty} N_{i+N} q^{-i} \end{bmatrix} \cdot \left(\sum_{i=0}^{\infty} N_i q^{-i} \right)^{-1} \Psi(t) \tag{8}$$

As a receding horizon approach, only the first control input computed from Eq. 7 is actually implemented in the proposed approach and the whole procedure is repeated at the next time step.

The formulation of the proposed method is the same as the one used in the MPC formulation except that optimal disturbance prediction is used in the proposed algorithm, as opposed to the step or random-walk-type disturbance model assumption in the MPC algorithm. If the disturbance model is actually a random-walk type one, the proposed algorithm becomes identical to the MPC algorithm.

Finite-Horizon Minimum Variance Controller

It is shown here that the first control input computed from Eq. 7 is the same as the one computed with a finite-horizon prediction. This enables one to design a moving horizon minimum variance controller with finite-horizon predictions. Before we present the main result of the present discussion, we will first state the following two lemmas. The first lemma comes from a direct application of the result given in Ko and Edgar (2001).

Lemma 1. The following identity holds for every nonnegative integer k.

$$\begin{aligned} & I - G_{d+k} \left(G_{d+k}^T \bar{Q}_{d+k} G_{d+k} \right)^{\dagger} G_{d+k}^T \bar{Q}_{d+k} \\ &= \begin{bmatrix} I - G_{d-1} \left(G_{d-1}^T \bar{Q}_{d-1} G_{d-1} \right)^{\dagger} G_{d-1}^T \bar{Q}_{d-1} & 0 \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{9}$$

where d is the delay order of the process, that is, the fewest number of time intervals for each output to be driven to any desired level by control actions.

Proof. See Ko and Edgar (2001).

The next lemma concerns the generalized inverse of partitioned matrices that was derived by Furuta and Wongsaisuwan (1993).

Lemma 2. Consider a partitioned symmetric matrix

$$H = \begin{bmatrix} L & M \\ M^T & N \end{bmatrix}$$

where the matrices M and N satisfy $M = MN^{\dagger}N$. Then, the generalized inverse of H is given by

$$H^{\dagger} = \begin{bmatrix} I & 0 \\ -N^{\dagger}M^T & I \end{bmatrix} \begin{bmatrix} (L - MN^{\dagger}M^T)^{\dagger} & 0 \\ 0 & N^{\dagger} \end{bmatrix} \begin{bmatrix} I & -MN^{\dagger} \\ 0 & I \end{bmatrix} \tag{10}$$

Proof. See Furuta and Wongsaisuwan (1993).

We can now state the main result of this section.

Theorem 1. The first control input $u(t)$ computed from

$$\bar{u}_d = \begin{bmatrix} u(t) \\ u(t+1) \\ \vdots \\ u(t+d-1) \end{bmatrix} = -\left(G_d^T \bar{Q}_d G_d\right)^{\dagger} G_d^T \bar{Q}_d \left(\tilde{y}_d + \tilde{\Psi}_d\right) \quad (11)$$

is equal to that from $\bar{u}_{d+k}$ for any positive integer k. Therefore, the first control input $u(t)$ associated with infinite-horizon can be conveniently computed from Eq. 11.

Proof. In this proof we will explicitly solve for the first control input in both $\bar{u}_d$ and $\bar{u}_{d+k}$, $(k>0)$. To this end, we write the block-Toeplitz matrix G_d in its partitioned form as

$$G_d = \begin{bmatrix} H_1 & 0 \\ \tilde{H}_{d-1} & G_{d-1} \end{bmatrix} \quad (12)$$

where $\tilde{H}_{d-1} = [H_2^T \quad H_3^T \ \cdots \ H_d^T]^T$. Then,

$$\bar{u}_d = -\left(G_d^T \bar{Q}_d G_d\right)^{\dagger} G_d^T \bar{Q}_d \left(\tilde{y}_d + \tilde{\Psi}_d\right)$$

$$= -\begin{bmatrix} H_1^T Q H_1 + \tilde{H}_{d-1}^T \bar{Q}_{d-1} \tilde{H}_{d-1} & \tilde{H}_{d-1}^T \bar{Q}_{d-1} G_{d-1} \\ G_{d-1}^T \bar{Q}_{d-1} \tilde{H}_{d-1} & G_{d-1}^T \bar{Q}_{d-1} G_{d-1} \end{bmatrix}^{\dagger}$$

$$\times \begin{bmatrix} H_1^T Q & \tilde{H}_{d-1}^T \bar{Q}_{d-1} \\ 0 & G_{d-1}^T \bar{Q}_{d-1} \end{bmatrix} \cdot \left(\tilde{y}_d + \tilde{\Psi}_d\right) \quad (13)$$

Applying the result in lemma 2 to the righthand side of Eq. 13 we obtain the expression for the first control input as

$$u(t) = \Big[H_1^T Q H_1 + \tilde{H}_{d-1}^T \bar{Q}_{d-1}\Big(I - G_{d-1}\left(G_{d-1}^T \bar{Q}_{d-1} G_{d-1}\right)^{\dagger}$$

$$\times G_{d-1}^T \bar{Q}_{d-1}\Big) \tilde{H}_{d-1}\Big]^{\dagger} \Big[H_1^T Q, \quad \tilde{H}_{d-1}^T \bar{Q}_{d-1}(I - G_{d-1}$$

$$\times \left(G_{d-1}^T \bar{Q}_{d-1} G_{d-1}\right)^{\dagger} G_{d-1}^T \bar{Q}_{d-1}\Big)\Big] \cdot \left(\tilde{y}_d + \tilde{\Psi}_d\right). \quad (14)$$

On the other hand, when the horizon length is $d+k$, we similarly partition the block-Toeplitz matrix G_{d+k} as

$$G_{d+k} = \begin{bmatrix} H_1 & 0 \\ \tilde{H}_{d+k-1} & G_{d+k-1} \end{bmatrix} \quad (15)$$

Then,

$$u_{d+k} = -\left(G_{d+k}^T \bar{Q}_{d+k} G_{d+k}\right)^{\dagger} G_{d+k}^T \bar{Q}_{d+k}\left(\tilde{y}_{d+k} + \tilde{\Psi}_{d+k}\right)$$

$$= -\begin{bmatrix} H_1^T Q H_1 + \tilde{H}_{d+k-1}^T \bar{Q}_{d+k-1} \tilde{H}_{d+k-1} & \tilde{H}_{d+k-1}^T \bar{Q}_{d+k-1} G_{d+k-1} \\ G_{d+k-1}^T \bar{Q}_{d+k-1} \tilde{H}_{d+k-1} & G_{d+k-1}^T \bar{Q}_{d+k-1} G_{d+k-1} \end{bmatrix}^{\dagger} \begin{bmatrix} H_1^T Q & \tilde{H}_{d+k-1}^T \bar{Q}_{d+k-1} \\ 0 & G_{d+k-1}^T \bar{Q}_{d+k-1} \end{bmatrix} \cdot \left(\tilde{y}_{d+k} + \tilde{\Psi}_{d+k}\right) \quad (16)$$

Applying lemma 2 to the righthand side of Eq. 16, we obtain

$$u(t) = \Big[H_1^T Q H_1 + \tilde{H}_{d+k-1}^T \bar{Q}_{d+k-1}(I - G_{d+k-1}$$

$$\times \left(G_{d+k-1}^T \bar{Q}_{d+k-1} G_{d+k-1}\right)^{\dagger} G_{d+k-1}^T \bar{Q}_{d+k-1}\Big) \tilde{H}_{d+k-1}\Big]^{\dagger}$$

$$\times \Big[H_1^T Q, \quad \tilde{H}_{d+k-1}^T \bar{Q}_{d+k-1}(I - G_{d+k-1}$$

$$\times \left(G_{d+k-1}^T \bar{Q}_{d+k-1} G_{d+k-1}\right)^{\dagger} G_{d+k-1}^T \bar{Q}_{d+k-1}\Big)\Big]$$

$$\cdot \left(\tilde{y}_{d+k} + \tilde{\Psi}_{d+k}\right) \quad (17)$$

Using lemma 1, it follows that

$$\tilde{H}_{d+k-1}^T \bar{Q}_{d+k-1}\Big(I - G_{d+k-1}\left(G_{d+k-1}^T \bar{Q}_{d+k-1} G_{d+k-1}\right)^{\dagger}$$

$$\times G_{d+k-1}^T \bar{Q}_{d+k-1}\Big) = \left[\tilde{H}_{d-1}^T, \quad * \right] \begin{bmatrix} Q_{d-1} & 0 \\ 0 & Q_k \end{bmatrix}$$

$$\times \begin{bmatrix} I - G_{d-1}\left(G_{d-1}^T \bar{Q}_{d-1} G_{d-1}\right)^{\dagger} G_{d-1}^T \bar{Q}_{d-1} & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \Big[\tilde{H}_{d-1}^T \bar{Q}_{d-1}(I - G_{d-1}$$

$$\times \left(G_{d-1}^T \bar{Q}_{d-1} G_{d-1}\right)^{\dagger} G_{d-1}^T \bar{Q}_{d-1}\Big) \tilde{H}_{d-1}, \quad 0\Big] \quad (18)$$

Substituting Eq. 18 into Eq. 17, the expression in Eq. 17 is reduced to that in Eq. 14.

Constrained Minimum Variance Controller

It is well known that the receding horizon approach of model predictive control provides a very flexible way to incorporate hard constraints on process variables into the computation of optimal control inputs (Garcia and Morshedi, 1986; Garcia et al., 1989; Muske and Rawlings, 1993; Qin and Badgwell, 1997). Since the design of the minimum variance controller proposed earlier is also based on the receding horizon concept, the same method of constrained optimization employed in MPC formulation can be used for the design of a constrained minimum variance controller. Conceptually, the constrained optimization should be carried out with infinite-horizon predictions to obtain a constrained minimumvariance controller. In most practical applications, however, the first control input computed will rapidly converge to their steady-state value as the horizon increases. Note, that in the unconstrained case, the convergence occurs immediately after the time step that is equal to the delay order of the process. If a sequence of control inputs and the corresponding outputs computed under a given horizon converge to their steady-state and set point values, respectively, within the given horizon, then further increments of horizon would not change the first control input computed. This can be used to check if the horizon used in the computation of control inputs is large enough to obtain the converged values of control inputs. In the following example, closed-loop behavior under a constrained minimum variance controller is examined in a 2×2 multivariable system by comparing it with an unconstrained minimum variance controller and an MPC controller.

Example 1

We consider the following 2×2 process model from an industrial fractionation column (Harris et al., 1996)

$$G(q^{-1}) = \begin{bmatrix} \dfrac{-1.50q^{-3}}{1-0.659q^{-1}} & \dfrac{-0.167q^{-1}}{1-0.923q^{-1}} \\ \dfrac{-0.519q^{-5}}{1-0.784q^{-1}} & \dfrac{0.154q^{-3}+0.144q^{-4}}{1-0.874q^{-1}} \end{bmatrix}. \quad (19)$$

The disturbance that enters at the process output is assumed to be described by the following model

$$\Psi(t) = \frac{1}{(1-q^{-1})(1-0.3q^{-1})} \begin{bmatrix} 1-0.5q^{-1} & -0.2q^{-1} \\ -0.5q^{-1} & 1-0.8q^{-1} \end{bmatrix} \cdot a(t) \quad (20)$$

Note that the disturbance model does not belong to a random-walk type class.

For a clear presentation of the closed-loop behavior, only a single noise vector $a(t) = [1 \ 0]^T$ is introduced at $t = 5$, and load disturbance responses are obtained with constrained and unconstrained minimum variance controllers and also with an unconstrained MPC controller. For all cases, the two outputs are assumed to have equal importance. The control inputs were constrained between ± 1.5 for the constrained minimum variance controller. For the unconstrained MPC controller, the control and the prediction horizons are all set to five (delay order of the process), and a zero move suppression coefficient is used.

The results of simulation are plotted in Figure 1. It is seen from this figure that the unconstrained minimum variance controller achieves faster disturbance rejection in y_1 using high initial control actions in u_2, while the constrained minimum variance controller results in a slower response in y_1 due to a constraint on u_2. The MPC controller in this case causes quite oscillatory behavior in the outputs with ringing occurring in both control inputs. The ringing behavior in the MPC controller can be attributed to a transmission zero of the process model that is located at -0.701. For minimum variance controllers, however, the ringing did not occur because the use of correct disturbance model in the minimum variance control resulted in controller zeros that tend to cancel out the ringing behavior associated with the process transmission zero at -0.701. Note that, under the constrained and the unconstrained minimum variance controllers, both outputs returned to their set points within five sampling intervals.

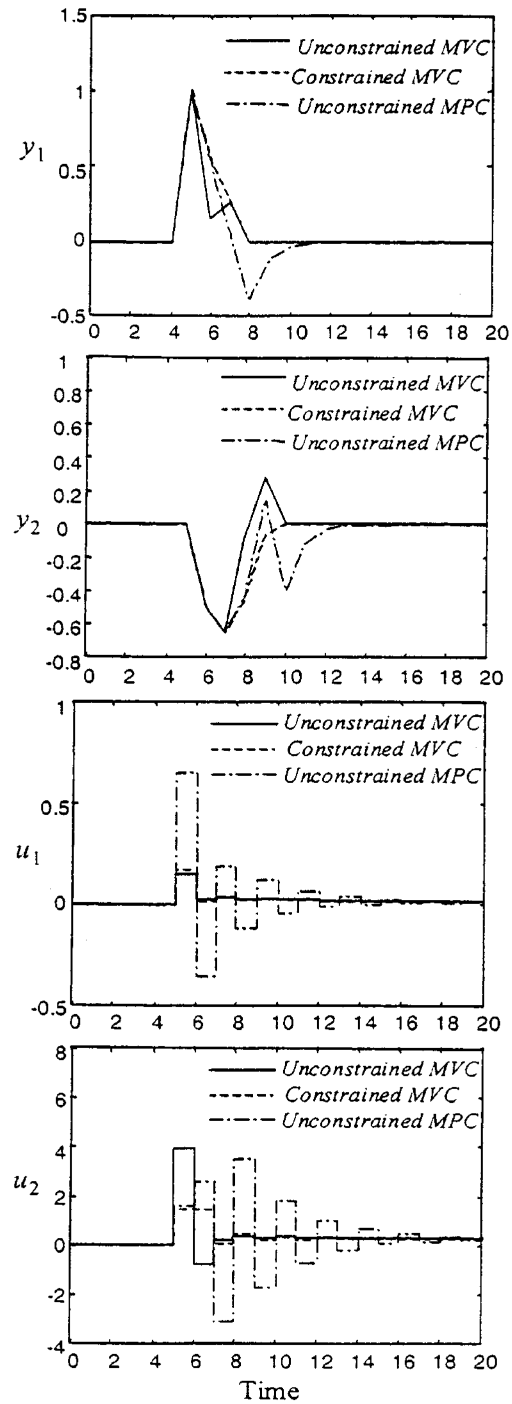

Figure 1. Closed-loop behavior for single load disturbance in example 1.

Performance Assessment of Constrained Model Predictive Control Systems

We describe a procedure for the estimation of achievable minimum variance performance bounds in constrained model predictive control systems via disturbance model identification and subsequent (constrained) closed-loop simulation with the constrained minimum variance controller developed earlier. In addition to the consideration of performance limitation due to the time-delay structure of the process, the procedure can illuminate the performance limitation due to hard constraints on process variables which might be a major cause degrading achievable performance.

The performance assessment of the constrained model predictive control system is done with regard to a sampled past data window. With an available process model, the disturbance sequence that entered at the process outputs can be estimated from the difference between the actual outputs and the model outputs. The disturbance sequence estimated in this way is then fitted into a time-series model to obtain a suitable disturbance model. For this purpose, an autoregressive model formulation can be advantageous in terms of its computational speed and capability of being recursively calculated (Graupe, 1989; Desborough and Harris, 1993). In addition to these advantages, an autoregressive model formulation can simplify the task of minimum mean-square error prediction of disturbance as shown here. When the disturbances exhibit nonstationary behavior, the difference between successive values of the disturbance sequence can be fitted into an autoregressive model structure resulting in an integrated autoregressive model for the disturbances.

Let the autoregressive disturbance model identified in this way be denoted as

$$\Psi(t) = \Phi_1\Psi(t-1) + \Phi_2\Psi(t-2) + \cdots + \Phi_m\Psi(t-m) + a(t) \tag{21}$$

where Φ_i is an autoregressive model coefficient matrice and m is the order of the model. The minimum mean-square error prediction of the disturbances can then be computed in a straightforward manner as follows (Åström, 1970; Box and Jenkins, 1970)

$$\begin{aligned}
\hat{\Psi}(t+1|t) &= \Phi_1\Psi(t) + \Phi_2\Psi(t-1) + \cdots + \Phi_m\Psi(t-m+1) \\
\hat{\Psi}(t+2|t) &= \Phi_1\hat{\Psi}(t+1|t) + \Phi_2\Psi(t) + \cdots + \Phi_m\Psi(t-m+2) \\
&\vdots \\
\hat{\Psi}(t+m+1|t) &= \Phi_1\hat{\Psi}(t+m|t) + \Phi_2\hat{\Psi}(t+m-1|t) \\
&\quad + \cdots + \Phi_m\hat{\Psi}(t+1|t) \\
&\vdots
\end{aligned} \tag{22}$$

where $\hat{\Psi}(t+i|t)$ is the minimum mean-square error forecast of $\Psi(t+i)$ given information available up to, and including time t. With these minimum mean-square error forecasts of disturbances, the matrix $\tilde{\Psi}_N$ given in Eq. 4 can be obtained from

$$\tilde{\Psi}_N = \begin{bmatrix} \hat{\Psi}(t+1|t) \\ \hat{\Psi}(t+2|t) \\ \vdots \\ \hat{\Psi}(t+N|t) \end{bmatrix}. \tag{23}$$

A constrained minimum variance controller can then be designed with these optimal disturbance predictions in the framework of receding horizon control as already described. The constrained minimum variance performance bound is finally estimated as the achieved performance of simulated outputs using a constrained minimum variance controller.

In calculating performance assessment of constrained model predictive control systems, a pre-assessment of achievable performance with an unconstrained minimum variance controller can be beneficial. In this pre-assessment, one can gain an insight into the significance of constraints on output performance, that is, the frequency and the size of control actions needed beyond the constraints. If there is no constraint violation in this pre-assessment step, the performance bound will then be computed simply from the output performance of this pre-assessment step using an unconstrained minimum variance controller. Otherwise, the effects of constraints on the variability of process outputs can be analyzed via a closed-loop simulation with the constrained minimum variance controller that employs optimal disturbance prediction and constrained optimization.

Example

In this section, we examine various aspects involved in the performance assessment of the constrained model predictive control system. Specifically, we investigate (1) the role of pre-assessment that employs an unconstrained minimum variance controller, (2) closed-loop behavior under a constrained minimum variance controller, (3) the convergence of calculated performance bounds with the horizon in constrained simulation, and (4), the effects of constraints on achievable performance bounds.

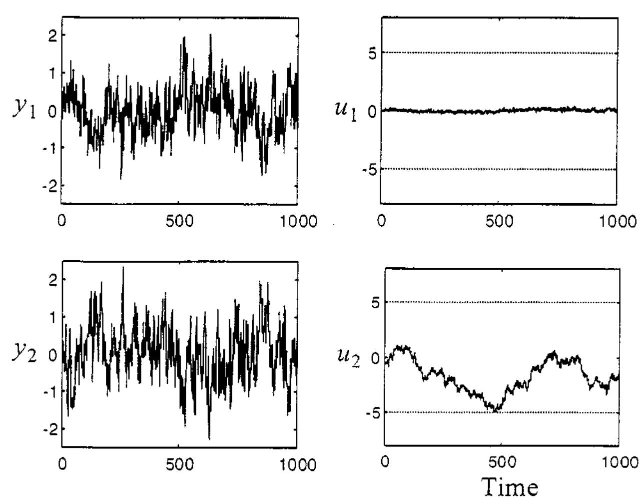

Figure 2. Closed-loop behavior with an MPC controller.

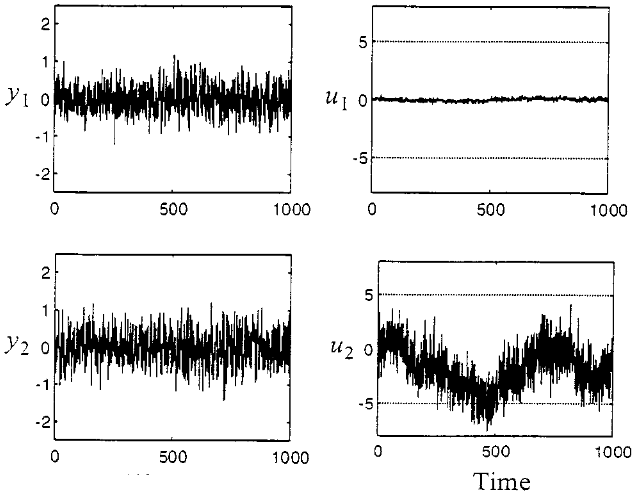

Figure 3. Closed-loop behavior with an unconstrained minimum variance controller.

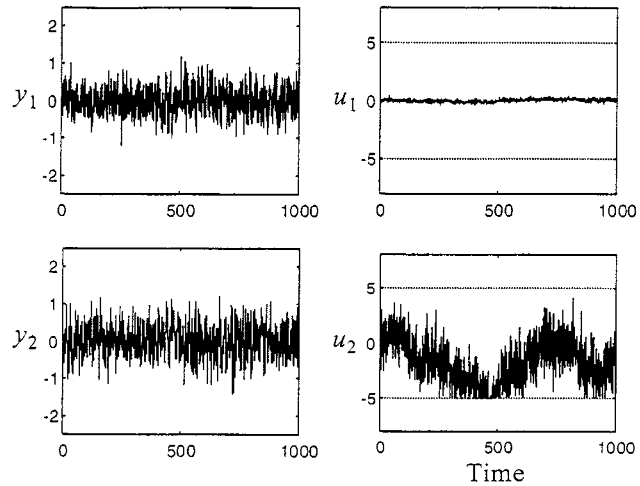

Figure 4. Closed-loop behavior with a constrained minimum variance controller.

We consider the same process and the disturbance models used in *Example 1*. As in *Example 1*, the two outputs are assumed to have equal importance. To regulate the outputs from load disturbances, an MPC controller was implemented that was tuned to have a prediction horizon of 20, a control horizon of one, and equal weightings on all the controlled and the manipulated variables, with the manipulated variables constrained between ± 5. Figure 2 shows simulated closed-loop behavior with this MPC controller. For this simulation, a sequence of random noise vectors $\{a(t)\}$ with the covariance matrix of $0.1I$ was used. The obtained output variances in this simulation were $E(y_1^2) = 0.380$, and $E(y_2^2) = 0.537$.

To examine the significance of input constraints on achievable output variance, we can perform a pre-assessment of achievable performance using an unconstrained minimum variance controller. In this optional pre-assessment step, we make an assumption that the hard constraints are not active. The unconstrained minimum variance controller in this case was designed based on a disturbance model that was identified as an integrated autoregressive model of order 10. The resulting closed-loop behavior is shown in Figure 3. The achieved output variances in this pre-assessment were obtained as $E(y_1^2) = 0.119$, and $E(y_2^2) = 0.193$. It is seen from Figure 3 that the control input u_2 violates the lower input limit persistently between 400 to 500 time intervals, while the control input u_1 stays within the limits during the whole time period. However, the control actions needed beyond the constraint limits and the frequency of constraint violations in u_2 are not so significant during the whole time period. Therefore, we can expect that the constrained minimum variance performance bounds will be close to those obtained in this pre-assessment step. It is worthwhile to note here, that the performance bounds obtained in the pre-assessment step can also be computed from other performance assessment procedures found in the literature (Harris et al., 1996; Huang et al., 1997; Ko and Edgar, 2001). For example, the procedure developed by Ko and Edgar (2001) results in achievable output variances as $E(y_1^2) = 0.121$ and $E(y_2^2) = 0.210$. Slight differences in achievable performance bounds with those of the pre-assessment step comes from the finite sampling effects in the time-series modeling of the outputs and the disturbances.

The constrained minimum variance performance bounds are then computed from a simulation using a constrained minimum variance controller. For the design of a constrained minimum variance controller the disturbance model identified in the pre-assessment step was used. The closed-loop behavior from this constrained simulation is plotted in Figure 4. In the constrained simulation a finite-horizon of 20 was used for the computation of optimal control inputs. In this figure we observe that the closed-loop behavior is very similar to that of the pre-assessment step except that the control input u_2 is constrained in its lower limit. In fact, the observed output variances were computed very close to those of the pre-assessment step, and they were $E(y_1^2) = 0.121$ and

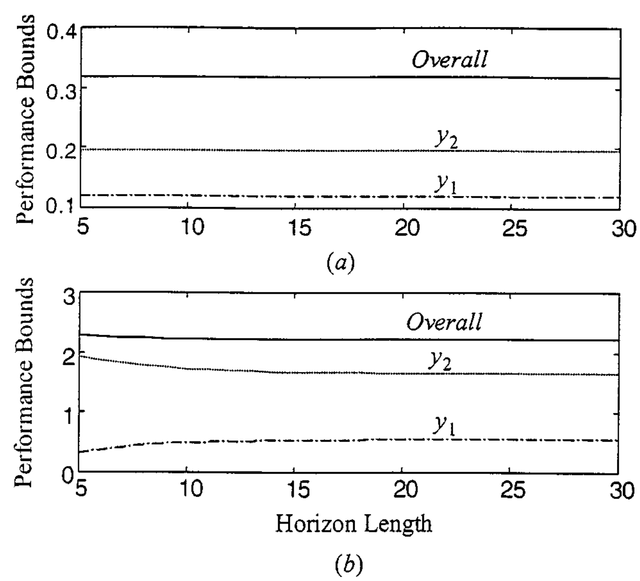

Figure 5. Convergence of calculated performance bounds with the horizon when control inputs are constrained between (a) ± 5 and (b) ± 3.

Table 1. Effects of Input Constraints on Achievable Performance Bounds

Limits on Control Inputs	Min. Variance Perform. on y_1	Min. Variance Perform. on y_2
± 1	5.132	17.208
± 2	1.864	6.110
± 3	0.559	1.664
± 4	0.169	0.356
± 5	0.121	0.197
± 6	0.119	0.193
± 7	0.119	0.193
± 8	0.119	0.193

$E(y_2^2) = 0.197$. Therefore we find that the effects of constraints are negligibly small for this example in the computation of minimum variance performance bounds.

To examine the effects of using a finite-horizon prediction in the computation of constrained minimum variance performance bounds, the constrained simulations were carried out with horizons ranging from 5 to 30. The resulting performance bounds were plotted in Figure 5a. It is observed in this figure that the performance bounds converge immediately after the horizon length of five. Therefore, it is confirmed that the performance bounds obtained in the previous constrained simulation with the horizon length of 20 were indeed converged ones. However, the convergence of performance bounds depends on the nature of constraints. When the constraints are changed so that the control inputs are constrained between ± 3, Figure 5b shows slightly slower convergence. As derived earlier, the convergence of performance bounds will occur immediately after the delay order of the process if the constraints are inactive.

It is now of interest to examine the effects of constraints on achievable performance bounds. Table 1 shows the performance bounds obtained for several different constraints on control inputs. From this table, we see that the unconstrained minimum variance performance bounds are recovered when the control inputs are constrained at ± 6 or higher. It is also seen from this table that the achievable output performance degrades significantly as the control inputs are constrained at ± 3 or smaller, and the achievable output variance of y_2 is more heavily affected by the changes in the input constraints.

Although, in this example, only the constraints on the magnitude of control inputs were considered, the proposed approach can easily be extended to include constraints on the outputs and the control input changes. In such a case, the significance of the effects of constraints on the outputs and the control input changes can be examined in addition to that of control input constraints during the pre-assessment step to properly assess the effects of various constraints. If any of these constraints are severely violated, the performance bounds cannot be determined from the pre-assessment step only. The constrained closed-loop simulation should be carried out to account for the effects of constraints.

The constrained minimum variance performance bounds estimated in this example may not be achievable by the traditional MPC controller where the step disturbance model is assumed. Depending on disturbances that are actually affecting the process, the MPC achievable performance bounds can degrade substantially from the constrained minimum variance performance bounds. However, the constrained minimum variance performance bounds represent the theoretical lower bounds on achievable performance in the constrained control systems, and, therefore, can be useful as a first-level benchmark performance.

Conclusions

A methodology for the performance assessment of a constrained model predictive control system has been developed in this article. This method assumes that the process model has a stable inverse. For this method, a constrained minimum variance controller was designed via optimal disturbance prediction and constrained optimization in the framework of receding horizon control. The constrained minimum variance performance bounds were then estimated from the simulation that employed the constrained minimum variance controller. When the constraints become inactive, the proposed method naturally recovered unconstrained minimum variance performance bounds.

In the estimation of constrained minimum variance performance bounds, a pre-assessment step was introduced that simulates closed-loop behavior using an unconstrained minimum variance controller, which was also developed in this article based on a finite-horizon output prediction. The pre-assessment step was found to be valuable for assessing the significance of constraints on output performance bounds.

The proposed method has been applied to an industrial fractionation column model in order to quantify the effects of input constraints on the minimum variance performance bounds.

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Appendix

From Eqs. 3 and 4 the quadratic cost function of actual outputs is

$$\lim_{N\to\infty}\frac{1}{N}E\left[\bar{y}_N^T\bar{Q}_N\bar{y}_N\right] \tag{A1}$$

where $\bar{y}_N=[y^T(t+1),\ y^T(t+2),\ \cdots,\ y^T(t+N)]^T$ and $\bar{Q}_N$ is an N-block diagonal matrix of Q. The minimum variance controller can be considered as a controller that minimizes the cost function in Eq. A1 at every discrete time t given observed outputs up to and including time t and all past control input signals. Since $\bar{a}_N$ is uncorrelated with $\bar{u}_N$, $\tilde{y}_N$, and $\tilde{\Psi}_N$ in Eq. 4 under a causal controller, Eq. A1 can be further written as

$$\lim_{N\to\infty}\frac{1}{N}E\left[\bar{y}_N^T\bar{Q}_N\bar{y}_N\right]$$
$$=\underbrace{\lim_{N\to\infty}\frac{1}{N}E\left[\left(G_N\bar{u}_N+\tilde{y}_N+\tilde{\Psi}_N\right)^T\bar{Q}_N\left(G_N\bar{u}_N+\tilde{y}_N+\tilde{\Psi}_N\right)\right]}_{\equiv E_1}$$
$$+\underbrace{\lim_{N\to\infty}\frac{1}{N}E\left[\left(D_N\bar{a}_N\right)^T\bar{Q}_N\left(D_N\bar{a}_N\right)\right]}_{\equiv E_2} \tag{A2}$$

The term E_2 in Eq. A2 is independent of any causal controller, and thus the cost function is minimized by minimizing the term E_1 in Eq. A2 which is the quadratic cost function of the minimum variance output prediction. Therefore, the receding horizon control that minimizes the quadratic cost function of minimum variance output prediction (E_1) leads to the minimum variance control. Note that the term E_2 in Eq. A2 can approach infinity when disturbances are nonstationary. It is because the cost function in Eq. A1 includes the quadratic term of future outputs while the controller can only use information available up to, and including, time t. However, this situation will not happen in practice as new updated control inputs will be implemented at every future time steps with new process measurements.

Manuscript received Jan. 24, 2000, and revision received Jan. 3, 2001.